Multihorizon Sharpe Ratios

It all depends on investment horizon.

Sangbae Kim and Francis In

Strategic asset allocation is a process that identifies the optimal portfolio for a given investor over a particular investment horizon. The holding period that is relevant is the length of time the investor holds any stocks or bonds, no matter how many changes are made among the individual issues in the portfolio (Siegel [1998]).

The Sharpe ratio in a mean-variance framework is a common measure of the performance of a portfolio. As Levy [1972] notes, the Sharpe ratio is closely related to the investment horizon. In other words, the horizon sensitivity of the Sharpe ratio is very important in evaluating portfolio performance of one or more portfolios.

Institutional investors such as pension funds have a very long investment horizon and are not necessarily interested in short-term performance. Yet a common practice in measuring the n-period Sharpe ratio is to use the expected return and standard deviation of a one-period investment by taking scaling factor into account (see Levy [1972]). This n-period Sharpe ratio is a complex and non-linear function of the expected return and standard deviation of a one-period investment. Scaling of this sort is valid only if the underlying data are identically and independently distributed, which may not be the case for financial time series (see Hodges, Taylor, and Yoder [1997]).

Consider an investment company with many investors and money managers. The investors and the money managers make decisions over different time periods. Suppose the investment horizon of an investor is one

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year, and that the investment company reviews the performance of the money manager every quarter, using the Sharpe ratio. The money manager is going to focus on the three-month performance of a portfolio, and the investor on the one-year performance. Thus, for this investor, the money manager may not provide the best service. To provide the best service for diverse investors, the Sharpe ratio needs to be constructed over different investment horizons.

Given the importance of multihorizon measurement, it seems odd that there are many empirical studies of the Sharpe ratio (e.g., Pedersen and Satchell [2000]), but there is only one published study of the multihorizon Sharpe ratio. Hodges, Taylor, and Yoder [1997] examine the multiperiod Sharpe ratio using randomized historical data from 1926 through 1993. They reach the unusual conclusion that bonds outperform stocks over sufficiently long holding periods.

One possible reason there is scant literature on the multihorizon Sharpe ratio is that we have lacked the appropriate econometric tool. We apply a new approach—wavelet analysis—to investigate the multihorizon Sharpe ratio, which we see as a significant contribution to the literature. Wavelet analysis allows us to decompose the unconditional variance into different time scales so that we can observe which investment horizons are important contributors to the time series variance. We can also construct the local average of the portfolio returns over each scale. These properties provide an effective way to construct the multihorizon Sharpe ratio.

SHARPE RATIO

Sharpe [1994] shows the fundamental case for the Sharpe ratio. Suppose we have a portfolio, i, with a return, R_i, and a risk-free rate of interest, f, with return R_f. The Sharpe ratio can be defined by:

$$SR_i = \frac{\overline{R}_i - \overline{R}_f}{\sigma_i} \tag{1}$$

where SR_i is the Sharpe ratio for portfolio i, $\overline{R}_i$ is the mean return of the portfolio, $\overline{R}_f$ is the mean return on the risk-free rate of interest, and σ_i is the standard deviation of the portfolio return. This Sharpe ratio captures the expected excess return per unit of risk associated with the excess return. If investors have a number of portfolios, they can evaluate the performance of each portfolio on the basis of its Sharpe ratio.

It is well known that the Sharpe ratio is an appropriate measurement of portfolio performance when portfolio returns are normally distributed. Yet many financial time series do not follow a normal distribution. Leland [1999] and Stutzer [2000] discuss measures of performance when returns are not distributed normally. Kazemi, Mahdavi, and Schneeweis [2003]) derive an adjusted Sharpe ratio by transforming the portfolio return distribution so that it will match the distribution of a benchmark. This approach is useful because it does not require normality of the returns.

Suppose a benchmark, B, has a return, R_B. The adjusted Sharpe ratio for a portfolio i can be defined as follows:

$$ASR_i = \frac{\overline{R}_B - \overline{R}_f}{\sigma_B} + \frac{(1 + \overline{R}_f)(1 - P)}{\sigma_B} \tag{2}$$

where

$$P = \frac{E[F(1 + R_i)]}{1 + R_f}$$

and $F(1 + R_i)$ indicates a function that matches the distribution of the portfolio returns with the distribution of the benchmark returns. P is the current value of $F(1 + R_i)$; $E(\cdot)$ means expectation under the risk-neutral probability distribution.

Note that the Sharpe ratio of the benchmark is the same as the Sharpe ratio in Equation (1). In other words, the difference between the Sharpe ratio of the benchmark and the adjusted Sharpe ratio of the portfolio is the term $(1 + \overline{R}_f)(1 - P)/\sigma_B$. In the adjusted Sharpe ratio, the value of P plays a crucial role. If $P < 1$, for example, the portfolio is a better investment than the benchmark; if $P > 1$, the benchmark outperforms the portfolio.

Given this adjusted Sharpe ratio, the multihorizon Sharpe ratio can be constructed using the wavelet variance and local average at scale λ_j as follows:[1]

$$ASR_i^w(\lambda_j) = SR_B^w(\lambda_j) + \frac{(1 + \overline{R}_f(\lambda_j))(1 - P(\lambda_j))}{\sqrt{\tilde{v}_B^2(\lambda_j)}} \tag{3}$$

where

$$SR_B^w(\lambda_j) = \frac{\overline{R}_B(\lambda_j) - \overline{R}_f(\lambda_j)}{\sqrt{\tilde{v}_B^2(\lambda_j)}} \tag{4}$$

EXHIBIT 1
Basic Statistics—1926–2000

$x =$	Treasury Bills	Large-Company Stocks	Small-Company Stocks	Long-Term Bonds	Intermediate-Term Bonds
Mean	0.0031	0.0099	0.0111	0.0046	0.0044
Variance	0.0000	0.0032	0.0088	0.0005	0.0002
Skewness	0.9701	0.3556	-0.0935	0.8488	1.1160
Kurtosis	1.0696	9.9408	16.2429	5.5099	10.8134
JB	184.0626	3724.7028	9895.0459	1246.5272	4571.6975
	(0.0000)	(0.0000)	(0.0000)	(0.0000)	(0.0000)
LB(15) for x	11198.2844	53.9777	57.8563	48.0894	71.3821
	(0.0000)	(0.0000)	(0.0000)	(0.0000)	(0.0000)
LB(15) for x^2	8920.7639	485.1915	241.6108	329.4144	180.6913
	(0.0000)	(0.0000)	(0.0000)	(0.0000)	(0.0000)

Skewness and kurtosis are defined as $[E(R_t - \mu)]^3$ and $[E(R_t - \mu)]^4$, respectively, where μ is the sample mean. JB indicates the Jarque-Bera statistic. LB(n) is the Ljung-Box statistic for up to n lags, distributed as $\chi 2$ with n degrees of freedom. Significance levels are in parentheses.

and $\overline{R}_B(\lambda_j)$ and $\overline{R}_f(\lambda_j)$ are the mean values of the benchmark portfolio return and the risk-free rate at scale λ_j.

These mean values are calculated using scaling coefficients, following Gençay, Selçuk, and Whitcher [2003]. In this specification, SR_B^w and ASR_i^w indicate the wavelet multiscale Sharpe ratio of the benchmark portfolio and the adjusted wavelet multiscale Sharpe ratio of a portfolio, which can vary depending on the wavelet scales (i.e., investment horizons).

To compare the traditional Sharpe ratio and the Sharpe ratios at each scale, it is informative to consider the spectral representation theorem. By this theorem, the spectrum of original monthly return series, R_t, includes all frequencies between zero and 1/2 cycles, equivalent to the 0 to 2-month period in our data frequency. The wavelet coefficients, d_1, are associated with the frequencies in the interval [1/4, 1/2], equivalent to the 2- to 4-month period, while the wavelet coefficients d_2 and d_3 include frequencies [1/8, 1/4] and [1/16, 1/8]. Thus, the Sharpe ratios at different scales represent the performance measure of a portfolio at various frequencies (that is, at various time scales).

DATA AND BASIC STATISTICS

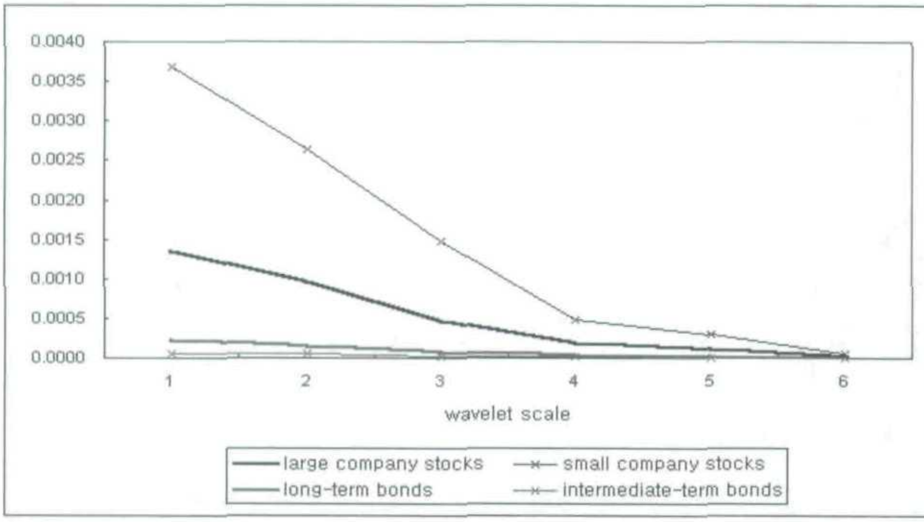
We use for data large- and small-company monthly stock returns and long-term and intermediate-term government bond returns, measured in nominal terms in *Stocks, Bonds, Bills, and Inflation* [2001]. The data periods range from January 1926 through December 2000.

Exhibit 1 presents summary statistics for the monthly data. Means are all positive and close to 0. The figures clearly show that stock returns are higher than bond returns. Another observation is that means and variances of the small-company stock returns are higher than those of the large-company stock returns, implying that the higher the return, the higher the risk.

Skewness, kurtosis, and Jarque-Bera statistics are also reported to check whether monthly data are normally distributed. The signs of the skewness vary, but the extent depends on the particular security. The Jarque-Bera statistics also indicate that all data are not normally distributed. The Ljung-Box (15) statistic for the squared return series is highly significant for all assets, suggesting the possible presence of autoregressive conditional heteroscedasticity. Notice that because of the non-normality of all returns, we use the adjusted Sharpe ratio instead of the traditional Sharpe ratio.

We want to examine the relationship between the Sharpe ratio and the investment horizon for portfolios of small-company stocks, large-company stocks, intermediate-term government bonds, and long-term government bonds. To investigate the multihorizon Sharpe ratio using wavelet analysis, we need to consider the balance between the sample size and the length of the wavelet filter. We settle on the Daubechies least asymmetric wavelet filter of length 8 [LA(8)], while we decompose our data up to scale 6. Since we use monthly data, scale 1 represents 2- to 4-month period dynamics. The scales 2, 3, 4, 5, and 6 correspond to 4- to 8-, 8- to 16-, 16- to 32-, 32- to 64,

Exhibit 2
Estimated Wavelet Variance



Multihorizon Sharpe ratios, using the large-company stocks as a benchmark, are presented in Exhibit 3. We see that the Sharpe ratio for each portfolio rises as the holding period lengthens. This result is consistent with Lo [2002]. The Sharpe ratio for the large-company stock portfolio is 18.08% for the first wavelet scale, equivalent to a 2- to 4-month period, and increases to 118.31% at the longest wavelet scale, equivalent to a 64- to 128-month period. A similar pattern is observed in the other asset returns.

These results indicate that the Sharpe ratio is not time-consistent. In other words, the trading strategy that leads to the most desirable portfolio for a 2- to 4-month period (scale 1) and for four consecutive 2- to 4-month periods is not the same as the strategy that gives the maximum Sharpe ratio for an 8- to 16-month period (scale 3). In other words, when the Sharpe ratio is used as a performance measure, we should take the horizon effect into account.

Exhibit 3 clarifies the relative rankings of the stock and bond portfolios over the wavelet scales. The Sharpe ratios for the stock portfolios are mostly higher than for the bond portfolios, except for scales 3 and 6 (8- to 16- and 64- to 128-month periods). Overall, our results are different from the results in Hodges, Taylor, and Yoder [1997], who find that the Sharpe ratio eventually declines in all cases and that bonds become more attractive than stocks for long holding periods.

Our results all the same mostly support the conventional wisdom of money managers that investors with long horizons should allocate a greater share of portfolio assets to stocks (Siegel [1998]). This result has to do with the mean-reversion property of stock returns.

As indicated in Samuelson [1991], mean reversion should encourage increased stock holdings if investors have a risk aversion coefficient greater than unity. Barberis [2000] finds that mean reversion in returns slows the growth of conditional variances of long-horizon returns. This makes equities appear less risky at long horizons, and hence more attractive to the investor.

Our results indicate that wavelet variance for short scales is higher than at longer scales, implying that an investor with a short investment horizon has to respond

and 64- to 128-month period dynamics, respectively.

First, we examine the variances of the four returns against various time scales. An important characteristic of the wavelet transform is its ability to decompose (analyze) the variance of the stochastic process. Exhibit 2 shows a pattern similar to that found by Siegel [1998, p. 32]. He argues that the standard deviation of average returns falls with the square root of the length of the holding period, due to the random walk of the asset returns.[2]

Exhibit 2 illustrates the wavelet variance of four series against the wavelet scales. The variances of both stock and bond markets decline as the wavelet scale increases. Note that the variance-versus-wavelet scale curves peak at the lowest scale (scale 1) in both markets.

A wavelet variance in a particular time scale indicates the contribution to sample variance (see In and Kim [forthcoming 2006]). The sample variances are 0.0099 for large-company stocks, 0.0111 for small-company stocks, 0.0046 for long-term bonds, and 0.0044 for intermediate-term bonds, and 43.6%, 42.7%, 45.5%, and 41.3% (respectively) of the total variance of the four portfolio returns is accounted for by the lowest scale (scale 1).[3]

This result implies that an investor with a short investment horizon must respond to every fluctuation in realized returns, while for an investor with a much longer horizon, the long-run risk is significantly lower. Overall, returns on stocks are more volatile than returns on bonds for all wavelet scales. Small-company stock returns are the most volatile among the four series. Intermediate government bond returns are the least volatile.

EXHIBIT 3
Multihorizon Sharpe Ratio

Wavelet Scale	Large-Company Stocks	Small-Company Stocks	Long-Term Bonds	Intermediate-Term Bonds
Scale 1	18.08%	18.09%	18.05%	17.99%
Scale 2	21.31%	21.59%	21.30%	21.26%
Scale 3	30.10%	30.10%	30.29%	30.31%
Scale 4	49.09%	50.27%	48.32%	48.59%
Scale 5	57.32%	57.09%	54.97%	54.86%
Scale 6	118.31%	120.61%	119.73%	120.99%

Scales 1, 2, 3, 4, 5, and 6 represent 2- to 4-, 4- to 8-, 8- to 16-, 16- to 32-, 32- to 64-, and 64- to 128-month period dynamics.

to every fluctuation in realized returns, while an investor with a much longer horizon faces much less significant long-run risk associated with unknown expected returns than short-run risk. Sharpe ratios at the long scale are much higher than at the short scale, suggesting that to use the Sharpe ratio as a performance measure, the horizon effect should be taken into account.

Finally, Sharpe ratios are higher for stock portfolios than for long-term and intermediate-term government bond portfolios except for scales 3 and 6, indicating that evaluation of the performance of stock and bond portfolios depends on the investment horizon. Therefore, in order for a money manager to provide the best service to investors, a multihorizon evaluation of performance is essential.

CONCLUDING REMARKS

Despite its importance in modern financial analysis, application of the Sharpe ratio has not been accompanied by examination of the investment horizon, an important factor for investors. We examine the multihorizon Sharpe ratio using wavelet analysis. It enables us to decompose a time series over various time scales so we can investigate the behavior of our data over multiple horizons.

Our results indicate that the wavelet variances of both the stock and bond markets decline as the wavelet scale increases, implying that an investor with a short investment horizon must be aware of every fluctuation in realized returns, while an investor with a much longer horizon faces less uncertainty with regard to unknown expected returns. In addition, Sharpe ratios at long scales are much higher than at short scales, implying that the Sharpe ratio is not time-consistent; the horizon should be taken into account.

Our results are different from the results in Hodges, Taylor, and Yoder [1997], who find the Sharpe ratio eventually declines in all cases and that, counterintuitively, bonds become more attractive than stocks for long holding periods. Sharpe ratios for stock portfolios are higher than for our bond portfolios except for some scales, indicating that performance of stock and bond portfolios depends on the investment horizon. Equities appear less risky at long horizons, and hence more attractive to the investor.

Our results generally support the conventional wisdom of money managers. Investors with long horizons, depending on the wavelet scale, should allocate a greater share of portfolio assets to stocks. A multihorizon evaluation of performance will enable money managers to provide more useful information to investors.

APPENDIX

Introduction to Wavelet Analysis

Wavelet analysis is a new approach to studying the multihorizon Sharpe ratio. One way to think about wavelets is in terms of wavelet and scaling filters. The relationship of filter banks and wavelets is discussed in Strang and Nguyen [1996] and Percival and Walden [2000].[4]

Like conventional Fourier analysis, wavelet analysis involves the projection of a signal onto an orthogonal set of components—sine and cosine functions in the case of Fourier analysis, and wavelets in the case of wavelet analysis. Wavelet analysis enables us to decompose data into several time scales.

To show the multiscale decomposition of a portfolio return series by wavelet analysis briefly, we use a simple Haar wavelet, which is the first wavelet filter. The Haar wavelet filter coefficient vector, of length $L = 2$, is given by $h = (h_0, h_1) = (1/\sqrt{2}, -1\sqrt{2})$. The complementary filter to h is the Haar scaling filter $g = (g_0, g_1) = (1/\sqrt{2}, 1/\sqrt{2})$. These filters have attributes as follows:

$$\sum_l h_l = 0, \sum_l h_l^2 = 1, \text{ and } \sum_l h_l h_{l+2n} = 0 \text{ for all integers } n \neq 0 \qquad \text{(A-1)}$$

$$\sum_l g_l = \sqrt{2}, \sum_l g_l^2 = 1, \text{ and } \sum_l g_l g_{l+2n} = 0 \text{ for all integers } n \neq 0 \quad \text{(A-2)}$$

Equation (A-1) indicates the three basic properties of wavelet filters: 1) They sum to zero, (2) they have unit energy, and (3) they are orthogonal to even shifts. The scaling filter in Equation (A-2) follows the same orthonormality properties of the wavelet filter, unit energy and orthogonality to even shifts, but instead of differencing consecutive blocks of observations, the scaling filter averages them. Thus, g may be viewed as a local averaging operator. Gençay, Selcuk, and Whitcher [2002] provide additional information regarding wavelet and scaling filters (including the Haar and longer compactly supported orthogonal wavelets) and their properties.

Using the Haar wavelet filter coefficients h, when applied to a return series R_t, the wavelet coefficients can be obtained by:

$$\sqrt{2}d_{1t} = h_0 R_t + h_1 R_{t-1}, t = 0, 1, \ldots, T-1 \quad \text{(A-3)}$$

The factor of $\sqrt{2}$ is required to ensure that the squared norm of the wavelet coefficients is equivalent to the squared norm of the return series. From Equation (A-3), one can see that the wavelet coefficient d_{1t} is a weighted difference between consecutive returns. In a similar manner, the scaling coefficients can be obtained using the Haar scaling filter coefficient vector g as follows:

$$\sqrt{2}s_{1t} = g_0 R_t + g_1 R_{t-1}, t = 0, 1, \ldots, T-1 \quad \text{(A-4)}$$

The scaling coefficients s_1 are based on local averages (of length two) of the original returns. Collecting both sets of coefficients into $w = (d_1, s_1)$ yields the high-frequency and low-frequency content of the original returns.

To derive the higher scale wavelet and scaling coefficients, it is necessary to calculate the higher scale wavelet and scaling filter coefficients using the first scale wavelet and scaling filters, h and g. To derive the scale 2 wavelet and scaling filters, let us define a filter $h' = (h_0, 0, h_1) = (1\sqrt{2}, 0, -1/\sqrt{2})$ to be the Haar wavelet filter with a zero between the two coefficients. The scale 2 Haar wavelet filter is calculated by:

$$h_{2,i} = \{g \times h_i'\} = \sum_{j=0}^{L-1} g_j h'_{i-j} \text{ for } i = 0, \ldots, 3 \quad \text{(A-5)}$$

From this equation, we can obtain the scale 2 wavelet filter, $h_2 = (1/2, 1/2, -1/2, -1/2)$ with length $L_2 = 4$. The wavelet coefficient d_2 may be obtained using h_2 via $2d_2 = \{h_2 \times R_t\}$ where the factor of 2 is a normalization constant. The scale 2 Haar wavelet filter first averages two pairs of returns and then proceeds to difference them. Thus, the wavelet coefficients d_2 are associated with changes on a scale of two. By defining $g' = (g_0, 0, g_1) = 1/\sqrt{2}, 0, 1/\sqrt{2})$, the scale 2 Haar scaling filter is obtained as follows:[5]

$$g_{2,i} = \{g \times g_i'\} = \sum_{j=0}^{L-1} g_j g'_{i-j} \text{ for } i = 0, \ldots, 3 \quad \text{(A-6)}$$

so that $g_2 = (1/2, 1/2, 1/2, 1/2)$. Note that g_2 is a simple average of four consecutive returns.

The scaling coefficient s_2 may be obtained directly using g_2 via $2s_2 = \{g_2 \times R_t\}$. Similarly, the scale 3 wavelet and scaling coefficients may be obtained by inserting increasing numbers of zeros between the wavelet and scaling filter coefficients such as h and g with length $L = 2$. For example, $h'' = (h_0, 0, 0, h_1) = (1/\sqrt{2}, 0, 0, -1\sqrt{2})$, and define the scale Haar wavelet filter to be $h_3 = [\{g \times g_i'\}h_i''] = \{g_2 \times h''\}$with the wavelet coefficient:

$$h_3 = \left(\frac{1}{\sqrt{8}}, \frac{1}{\sqrt{8}}, \frac{1}{\sqrt{8}}, \frac{1}{\sqrt{8}}, \frac{-1}{\sqrt{8}}, \frac{-1}{\sqrt{8}}, \frac{-1}{\sqrt{8}}, \frac{-1}{\sqrt{8}}\right)$$

with length $L_3 = 8$.

Generally, the length of a scale j wavelet filter can be calculated by $L_j = (2^j - 1)(L - 1) + 1$. Similarly, the scale 3 scaling filter coefficient is calculated as $g_3 = [\{g \times g_i'\}g_i''] = \{g_2 \times g_i''\}$ where $g'' = (g_0, 0, 0, g_1) = (1/\sqrt{2}, 0, 0, 1/\sqrt{2})$ with coefficient:

$$g_3 = \left(\frac{1}{\sqrt{8}}, \frac{1}{\sqrt{8}}, \frac{1}{\sqrt{8}}, \frac{1}{\sqrt{8}}, \frac{1}{\sqrt{8}}, \frac{1}{\sqrt{8}}, \frac{1}{\sqrt{8}}, \frac{1}{\sqrt{8}}\right)$$

Like the calculation of the scale 2 wavelet and scaling coefficients, the scale 3 wavelet and scaling coefficients are calculated via $\sqrt{8}d_3 = \{h_3 \times R_t\}$, and $\sqrt{8}s_3 = \{g_3 \times R_t\}$, respectively.

This procedure may be repeated up to the scale $J = \log2^T$ with the resulting wavelet and scaling coefficients organized into the vector $w = (d_1, d_2, \ldots, d_j, s_j)$. The wavelet coefficients from scale $j = 1, 2, \ldots, J$ are associated with the frequency interval $[1/2^{j+1}, 1/2^j]$, while the remaining scaling coefficients s_J are associated with the remaining frequencies $[0, 1/2^{j+1}]$. In this specification, the wavelet coefficients $d_J, \ldots, d_1$, which can capture the higher-frequency oscillations, represent increasingly fine scale deviations from the smooth trend, and s_J represents the smooth coefficients that capture the trend.

Using the wavelet coefficients, the wavelet variance for scale λ_j can be estimated by:

$$\tilde{v}^2(\lambda_j) \equiv Var(d_{jt}) \quad \text{(A-7)}$$

ENDNOTES

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[1]More detail about wavelet analysis and how to derive the wavelet variance is in the appendix.

[2]Levy and Gunthorpe [1993] show that under the iid assumption, the variance increases rapidly with lengthening of the holding period. It is of interest to compare Levy and Gunthorpe's [1993] result with Exhibit 2, even though Exhibit 1 shows that the asset returns do not follow the iid assumption. In many analyses, converting estimates from one frequency to another is necessary to yield a fair comparison. An approach using the n-period variance, as shown in Equation (2) in Levy and Gunthorpe [1993], may generate a different result. In the financial markets, all investors may

simultaneously operate on several time horizons so that the single-period variance includes all responses of investors with the different investment horizons. Mathematically, from the fundamental properties of the spectral density function of a stationary process, the single-period return spectrum spans zero to 1/2 cycle per unit (month, quarter, or year), implying that the single-period variance includes every fluctuation in all frequencies. Therefore, using the single-period variance to calculate the n-period variance may make the variance increase as the holding period lengthens. In wavelet analysis, however, one can isolate the variation of an asset return on a frequency-by-frequency (or scale-by-scale) basis, depending on investment horizons. Therefore, the wavelet variance examined here decomposes the variance of an asset return on a scale-by-scale basis. This produces different results from Levy and Gunthorpe [1993]. We thank a referee for pointing this out.

[3]Calculated by the normalization of wavelet variance using the sample variance. For more detail, see In and Kim [forthcoming 2006].

[4]According to Ramsey [2002], one can approach analysis of the properties of wavelets either through wavelets or through the properties of the filter banks.

[5]For the other filters, such as the Daubechies least asymmetric wavelet filter of length 8 [LA(8)], construction of the higher scale filters is similar. For instance, the scale 1 LA(8) scaling filter, g, is (-0.0758, -0.0296, 0.4976, 0.8037, 0.2979, -0.0992, -0.0126, 0.0322). In this case, is g' defined as (-0.0758, 0, -0.0296, 0, 0.4976, 0, 0.8037, 0, 0.2979, 0, -0.0992, 0, -0.0126, 0, 0.0322). For the various scaling filter coefficients, see Gençay, Selçuk, and Whitcher. [2002, pp. 114-115].

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